JOURNAL OF FUNCTIONAL ANALYSIS 67, 1–24 (1986)

On L^p Multipliers of Cartan Motion Groups

A. H. DOOLEY

School of Mathematics, University of New South Wales, Kensington, Australia 2033

AND

G. I. GAUDRY

School of Mathematical Sciences, Flinders University, Bedford Park, Australia 5042

Communicated by Ralph S. Phillips

Received April 10, 1983; revised May 20, 1985

Let (G, K) be a Riemannian symmetric pair of the compact type and let $V \rtimes K$ be the associated Cartan motion group. We establish a body of approximation theorems that permit the transfer of Fourier analysis from the group G to the Cartan motion group. As a consequence, we prove an analogue of a theorem of de Leeuw concerning restriction of Fourier multipliers of the group $\mathbb{R}$ to the group $\mathbb{Z}$. The latter theorem extends an earlier result of R. L. Rubin for the case where $G = SO(3)$, $K = SO(2)$, and $V \rtimes K$ is the Euclidean motion group. © 1986 Academic Press, Inc.

1. INTRODUCTION

The idea of considering the real line as a limit of a family of circles of increasing radii, and thus relating Fourier analysis on the line to that on the circle is an important technique of classical harmonic analysis. A significant application of this technique is to the theorems of deLeeuw [4], which relate L^p Fourier multipliers on $\mathbb{R}^m$ to those on $\mathbb{T}^m$. The second of his theorems states:

THEOREM A. *Suppose that* $1 \leqslant p < \infty$, *and that* Φ *is a bounded continuous function on* $\mathbb{R}^m (\cong \hat{\mathbb{R}}^m)$. *For each* $\lambda > 0$, *denote by* $\Phi^{(\lambda)}$ *the function on* $\mathbb{Z}^m (\cong \hat{\mathbb{T}}^m)$ *defined by the formula*

$$\Phi^{(\lambda)}(k_1, \dots, k_m) = \Phi\left(\frac{k_1}{\lambda}, \dots, \frac{k_m}{\lambda}\right).$$

0022-1236/86 $3.00

Suppose that for some sequence $\{\lambda_n\}$ such that $\lambda_n \to \infty$, each $\Phi^{(\lambda_n)}$ is a Fourier multiplier of $L^p(\mathbb{T})$, and that

$$\sup_n |||\Phi^{(\lambda_n)}|||_p = K < +\infty.$$

Then Φ is a Fourier multiplier of $L^p(\mathbb{R}^m)$ and

$$|||\Phi|||_p \leqslant K.$$

(The expression $|||\cdot|||_p$ denotes the norm of the operator on L^p induced by the multiplier in question.)

Analogues of this theorem, in the guise of transplantation theorems for various special function expansions, have been obtained by various authors (e.g., S. Igari [10], R. Strichartz [18], A. Bonami and J.-L. Clerc [1]) culminating in the work of Clerc [3] and of R. Stanton [17], who proved independently an analogue of Theorem A for compact symmetric spaces. Thus, let (G, K) be a Riemannian symmetric pair of the compact type [8, Chap. VII]. The role of $\mathbb{T}^m$ is played by G/K and the role of $\mathbb{R}^m$ by V, where $\mathbf{g} = \mathbf{k} + V$ is a Cartan decomposition of the semisimple Lie algebra $\mathbf{g}$. Let Λ denote the class of K-class 1 highest weights of $\mathbf{g}$; associate to each element β of Λ the element H_β of V such that $\beta(v) = \langle H_\beta, v\rangle$, $\langle\cdot,\cdot\rangle$ denoting the Killing form on $\mathbf{g}$. Theorem 2.5 of [17] or Théorème 2 of [3] can be stated as follows:

THEOREM B. *Let Φ be a bounded continuous* A$d(K)$*-invariant function on V, and p be fixed in the range $1 \leqslant p < \infty$. Suppose that for some sequence $\{\lambda_n\}$ such that $\lambda_n \to +\infty$, the functions $\Phi^{(\lambda_n)}$ defined on Λ by the formula*

$$\Phi^{(\lambda_n)}(\beta) = \Phi\left(\frac{H_\beta}{\lambda_n}\right)$$

are K-spherical multipliers of $L^p(G/K)$, and that their operator norms are uniformly bounded. Then Φ is a Fourier mutliplier of $L^p(V)$.

A further theorem of this type was proved by R. Rubin [15], who showed in the case where $G = SO(3)$, $K = SO(2)$, $V = \mathbb{R}^2$, how to relate L^p-Fourier multipliers on the Euclidean motion group $M(2)$ ($\cong \mathbb{R}^2 \rtimes SO(2)$) to Fourier mulipliers on $SO(3)$. Rubin's work is notable for being the first not to require any conditions of K-invariance on the multipliers.

The main result of the present article (Theorem 2.3) is a generalisation both of Rubin's theorem and of the Clerc–Stanton theorem. That is, we prove an analogue of deLeeuw's theorem in which the rôle of $\mathbb{T}^m$ is played by G and that of $\mathbb{R}^m$ by the semidirect product $V \rtimes K$. Here G is any compact Lie group having an analytic involution $\tilde{\theta}$ and a closed subgroup K

such that $(G_{\tilde{\theta}})_e \subseteq K \subseteq G_{\tilde{\theta}}$; $\mathbf{k}$ is the set of points in the Lie algebra $\mathbf{g}$ that are fixed by the involution $\theta = d\tilde{\theta}_e$; $G_{\tilde{\theta}}$ denotes the fixed-point group of $\tilde{\theta}$, and $(G_{\tilde{\theta}})_e$ its identity component.

The representation theories of the groups $V \rtimes K$ and G are of course well understood, and one knows how to compute their Plancherel measures. Consequently, there is an entirely natural notion of *Fourier multiplier* for each of the groups. A more substantial issue is how, starting with an (infinite-dimensional) operator-valued function Φ on the "dual" of $V \rtimes K$ one ought to define the associated family $\Phi(\lambda)$ of operator-valued functions on $\hat{G}$.

This is done by using techniques from [7]; Section 2 contains a summary of this preliminary material and a statement of the theorem: in Sections 3–5 we give the proof of the theorem.

In a previous paper [6], we established these results for the case $G = SO(n+1)$, $K = SO(n)$. The present results were announced in Section 4 of [5].

It is our hope that the techniques and ideas developed in this paper will find application to other problems of analysis on (particular cases of) Lie groups.

2. Preliminaries

In this section we will assemble the various prerequisites and give a statement of our main theorem.

(2.1) *Compact Symmetric Pairs*

Let (G, K) be a compact symmetric pair; thus G is a compact, connected Lie group (not necessarily semisimple) with involution $\tilde{\theta}$ and $(K^{\tilde{\theta}})_e \subseteq K \subseteq K^{\tilde{\theta}}$, where $K^{\tilde{\theta}}$ is the set of fixed points of θ. The basic properties of Riemannian symmetric pairs, where G is semisimple, are to be found in [8]; a nice exposition of the general case is to be found in Section 1 of [12]. We shall adopt notation and terminology from these two sources without further explanation.

Denoting $i\mathbf{p}$ by V, we have the Cartan decomposition $\mathbf{g} = \mathbf{k} + V$, and the Cartan motion group $V \rtimes K$ associated to G (where K acts by the adjoint action on V). We shall explore the relationship between analysis on G and harmonic analysis on $V \rtimes K$ by use of the contraction maps

$$\pi_\lambda : V \rtimes K \to G : (v, k) \to \exp_G \frac{v}{\lambda} . k \qquad (\lambda > 0),$$

A central tool in our proofs will be an approximation theorem for matrix coefficients, Theorem 2 of [7]. However, in [7] only the Riemannian case was considered, whereas we wish to consider the general case. For that reason we will give here a resumé of results of [7] in this slightly more general setting. Proofs are sufficiently similar to be omitted.

The irreducible representations of G may be described by an obvious modification of Section 5 of [7], itself a variant of the Borel–Weil theorem. Thus, we let $\mathbf{a}$ be a maximal abelian subalgebra of V, $A = \exp \mathbf{a}$, and let M be the centralizer of $\mathbf{a}$ in K. An irreducible representation of MA will have the form $\eta \otimes e^{i\phi}$, where $\eta \in \hat{M}$, $e^{i\phi} \in \hat{A}$ and for $m \in M \cap A$, $e^{i\phi(m)}\eta(m)$ is the identity. (Henceforth, when we write $\eta \otimes e^{i\phi} \in (MA)\hat{}$ we will assume this to be the case.) Associated to $\eta \otimes e^{i\phi}$ we have an irreducible representation $\sigma_{\phi,\eta}$ of G which acts by the left regular representation in the space $\mathbb{H}_{\phi,\eta}$. The latter is defined by (5.4) (xvi) and (xvii) of [7], where M and A are as above and P_+ is as in [12]. We note for future reference that if $e^{i\phi_0} \otimes \eta$ and $e^{i\phi} \otimes \eta$ both belong to $(MA)\hat{}$ then $\phi - \phi_0 \in \mathbf{a}$ and $e^{i(\phi - \phi_0)}$ is trivial on $M \cap A$. In particular, if $\phi > \phi_0$ this means hat $\phi - \phi_0 \in \mathscr{P}$, the lattice of K-class one weights of G.

The generic irreducible representation of $V \rtimes K$ are simply described, as in Section 3 of [7], by the Mackey orbital analysis. Thus, given $\eta \in \hat{M}$ and $\phi \in \mathbf{a}^{*+}$, the latter defined in (1.1) of [12], we have the irreducible representation $\rho_{\phi,\eta}$ of $V \rtimes K$ which acts in the space

$$\mathscr{H}^{\eta} = \{ f \in L^2(K, \mathscr{H}_\eta) \colon \eta(m) f(km) = f(k), \qquad \forall m \in M, k \in K \}$$

by

$$\rho_{\phi,\eta}(v, k) f(k_0) = e^{i\phi(\mathrm{Ad}(k_0^{-1})v)} f(k^{-1} k_0), \qquad v \in V, k, k_0 \in K.$$

We will denote by $X \subseteq (V \rtimes K)\hat{}$ the set of generic irreducible representations. It follows from Theorem 4.4 of [11] that X is a set of full Plancherel measure in $(V \rtimes K)\hat{}$.

(2.2) *The Approximation Theorem*

The main result of [7], suitably modified, may now be stated

(i) Suppose that $\eta \otimes e^{i\phi} \in (MA)\hat{}$. Denote by $\mathscr{R}_{\phi,\eta}$ the mapping $f \to f|_K$ for $f \in \mathscr{H}_{\phi,\eta}$. Then for all ϕ, $\mathscr{R}_{\phi,\eta} \mathscr{H}_{\phi,\eta} \subseteq \mathscr{H}^{\eta}$. Furthermore, this mapping is an injection.

(The second statement is not actually proved in [7]. It may be seen by using Matsuki's classification of the K^C orbits in G^C/B. For a simple proof, see [5, Sect. 4].)

At times, when $\sigma = \sigma_{\phi,\eta}$ we will find it convenient to write $\mathscr{R}_\sigma$ instead of $\mathscr{R}_{\phi,\eta}$.

(ii) Note that if s is a K-fixed vector in $\mathscr{H}_{\psi,1}$, $\psi \in \mathscr{P}$ with $s(k) = 1$ for all $k \in K$, then $\mathscr{R}_{\psi+\phi,\eta} s \cdot f = \mathscr{R}_{\psi,\eta} f$. This simple fact, combined with the injectivity of $\mathscr{R}_{\psi,\eta}$, allows us to write Theorem 2 of [7] in the following form.

THEOREM. *Let $e^{i\psi_0} \otimes \eta \in (MA)\hat{}$ and consider the family of representations $\sigma_{(\psi_0+\gamma),\eta}$, $\gamma \in \mathscr{P}$. Suppose that $f \in \mathscr{R}_{\psi_0,\gamma}\mathscr{H}_{\psi_0,\gamma}$, and that M_1, M_2, $M_3 > 0$.*

There exists a constant $c = c(M_1, M_2, M_3, f)$ such that whenever $\phi \in \mathbf{a}^{*+}$, $\|v\| \leqslant M_1$, $\|(\psi_0+\gamma)/\lambda\| \leqslant M_2$ and $\|\psi_0+\gamma-\lambda\phi\| \leqslant M_3$, we have

$$\|\rho_{\phi,\eta}(v,k)f - \mathscr{R}_{\psi_0+\gamma,\eta}\sigma_{\psi_0+\gamma,\eta}(\pi_\lambda(v,k))\mathscr{R}_{\psi_0+\gamma,\eta}^{-1}f\|_{\mathscr{H}^\eta} \leqslant \frac{c}{\lambda}.$$

In particular, for all $\psi \in \mathbf{a}^{*+}$ with $\psi - \psi_0 \in \mathscr{P}$,

$$\|\rho_{\psi/\lambda,\eta}(v,k)f - \mathscr{R}_{\psi,\eta}\sigma_{\psi,\eta}(\pi_\lambda(v,k))\mathscr{R}_{\psi,\eta}^{-1}f\|_{\mathscr{H}^\eta} \leqslant \frac{c}{\lambda}$$

whenever $\|v\| \leqslant M_1$ and $\|\psi\|/\lambda \leqslant M_2$.

(iii) The above theorem yields the following:

COROLLARY. *Let $e^{i\psi} \otimes \eta \in (MA)\hat{}$,*

(a) *Denoting by $\mathscr{H}_n$ the space $\mathscr{H}_{\psi+n\gamma,\eta}|_K$, one has*

$$\mathscr{H}_1 \subseteq \mathscr{H}_2 \subseteq \cdots \subseteq \mathscr{H}^\eta$$

and $\bigcup_j \mathscr{H}_j$ is dense in $\mathscr{H}^\eta$.

(b) *If γ_0 is any element of $\mathscr{P}$ then*

$$\mathscr{H}_{\psi+\gamma_0,\eta}|_K \subseteq \mathscr{H}_n$$

for all sufficiently large n, so that

$$\bigcup_{\gamma \in \mathscr{P}} \mathscr{R}_{\psi+\gamma,\eta}\mathscr{H}_{\psi+\gamma,\eta} \qquad \text{is dense in } \mathscr{H}^\eta$$

(2.3) *Statement of the Theorem*

DEFINITION. *A continuous operator-valued function Φ* on *X* is an assignment to every $\omega = (\psi, \eta) \in X$ of an operator $\Phi(\omega) \in \mathscr{B}(\mathscr{H}^\eta)$ such that for fixed η the function

$$\psi \to \Phi(\eta, \psi)$$

is continuous from $\mathbf{a}^{*+}$ into $(\mathscr{H}^\eta)$ with the strong operator topology.

Suppose that $1 \leqslant p < \infty$. A continuous operator-valued function Φ on X is said to be a *Fourier multiplier of* $L^p(V \rtimes K)$ if

(i) Φ is bounded, i.e.,

$$\sup_{\omega \in X} \|\Phi(\omega)\| < +\infty;$$

(ii) for every $f \in L^p \cap L^2(V \rtimes K)$, the function $\Phi\hat{f}$ defined a.e. on the dual of $V \rtimes K$ by the formula

$$\Phi\hat{f}(\omega) = \Phi(\omega)\,\hat{f}(\omega)$$

is the Fourier transform of a function in $L^p(V \rtimes K)$;

(iii) there is a constant M such that

$$\|\mathscr{F}^{-1}(\Phi\hat{f})\|_p \leqslant M\|f\|_p \tag{1}$$

for all $f \in L^p \cap L^2(V \rtimes K)$.

The smallest number M such that (1) holds is denoted $|||\Phi|||_p$, and is called the *p-multiplier norm* of Φ. The set of Fourier multipliers of $L^p(V \rtimes K)$ is denoted $M_p(V \rtimes K)$.

This definition is the natural analogue of the one applicable to compact groups given in [9], which we shall also use, with notations like those above. For example, we shall write $M_p(G)$ and $|||\cdot|||_p$, etc. when dealing with the compact group G.

Given a continuous operator-valued function Φ on X, and $\lambda > 0$, we define an operator-valued function $\Phi^{(\lambda)}$ on the dual of G as follows. If $e^{i\psi} \otimes \eta \in (MA)\hat{}$ let $\sigma = \sigma_{\psi,\eta}$. In the case where ψ is generic, we set

$$\Phi^{(\lambda)}(\sigma) = \mathscr{R}_\sigma^{-1}\mathscr{P}_\sigma \Phi\left(\eta, \frac{\psi}{\lambda}\right)\mathscr{R}_\sigma. \tag{2}$$

If ψ is not generic, we set $\Phi^{(\lambda)}(\sigma) = 0$.

In (2), $\mathscr{P}_\sigma$ denotes the projection mapping from $\mathscr{H}^n$ onto $\mathscr{R}_{\psi,\eta}\mathscr{H}_{\psi,\eta}$. As before, we will sometimes write $\mathscr{P}_{\psi,\eta}$ for $\mathscr{P}_{\sigma_{\psi,\eta}}$.

We can now state our version of the deLeeuw–Rubin theorem.

THEOREM. *Suppose that* $1 \leqslant p < +\infty$, *and that* Φ *is a bounded continuous operator-valued function on* X. *Suppose that for some sequence* $\{\lambda_n\}$ *of positive numbers tending to* $+\infty$, *the functions* $\Phi^{(\lambda_n)}$ *are Fourier multipliers of* $L^p(G)$ *for all* n *and that*

$$\limsup_{n \to +\infty} |||\Phi^{(\lambda_n)}|||_p = M < +\infty.$$

Then $\Phi \in M_p(V \rtimes K)$, *and* $|||\Phi|||_p \leqslant CM$, *where* C *is a constant determined by the groups* G *and* K.

Remark. The procedure for defining $\Phi^{(\lambda)}$ in (2) is to be compared with the corresponding definition given by Rubin [15], in which truncation of matrices is employed.

3. Approximation of Integrals

The mappings $\pi_\lambda (\lambda > 0)$ of $V \rtimes K$ into G induce mappings of functions f carried by $V \rtimes K$ to functions f_λ carried by G. The purpose of this section is to define the mapping $f \to f_\lambda$ and to derive an approximation result (3.3) concerning the integral f and the integrals of the functions f_λ as $\lambda \to +\infty$. We begin with a further result concerning the mapping π_λ.

(3.1) Lemma. *There exists a neighbourhood $\Omega \times K$ of $\{0\} \times K$ in $V \rtimes K$ such that*

(i) *Ω is* $\mathrm{Ad}(K)$*-invariant;*

(ii) *$\pi_1(\Omega \times K)$ is open;*

(iii) *π_1 is injective on $\Omega \times K$.*

The simple proof of this result is left to the reader.

(3.2) Definition. Suppose that $f \in C_c(V \rtimes K)$, and that f is supported in the set $B \times K$, B being compact. For all sufficiently large $\lambda > 0$, we have $B \subseteq \lambda\Omega$, so we can define a function f_λ on G by setting

$$f_\lambda(g) = \begin{cases} f \circ \pi_\lambda^{-1}(g) & \text{on} \quad \pi_\lambda(\lambda\Omega \times K), \\ 0 & \text{elsewhere.} \end{cases}$$

We come now to the result linking the integrals of f and of f_λ.

(3.3) Lemma. *Suppose $f \in C_c(V \rtimes K)$ and that $\lambda > 0$ is so large that f_λ is defined. Then*

(i) $$\int_G f_\lambda(g)\, dg = \mathrm{vol}(G/K)^{-1} \int_{\mathbf{a} \cap \Omega} \left|\prod_{\alpha \in P_+} \sin \alpha(iH)\right| \times \int_{K/M} \int_K f_\lambda(\exp[\mathrm{Ad}(k_1) H]\, k_2)\, dk_2\, d(k_1 M)\, dH;$$

(ii) *there is a constant C (determined by f) such that, for all sufficiently large λ*

$$\left| \lambda^{\dim V} \int_G f_\lambda(g)\, dg - \mathrm{vol}(G/K)^{-1} \int_{V \rtimes K} f(v, k)\, dv\, dk \right| \leqslant \frac{C}{\lambda^2}.$$

Proof. (i) As in [8], consider the mapping β: $(kM, H) \to \mathrm{Ad}(k) H$ of $(K/M) \times \mathbf{a}$ onto V(cf. [8, V.6.3]). Let ψ be the mapping

$$\psi = \mathrm{Exp}_{G/K} \circ \beta$$

of $(K/M)\times\mathbf{a}$ onto G/K. The Jacobian factor for the change of measure induced by ψ is given by the formula

$$\psi^*(d(gK))=\left|\prod_{\alpha\in P_+}\sin\alpha(iH)\right| d(kM)\,dH.$$

Moreover, $\operatorname{Exp}_{G/K}$ is the composition with exp of the quotient mapping of G onto G/K. So

$$\int_G f_\lambda(g)\,dg=\operatorname{vol}(G/K)^{-1}\int_{G/K}\left(\int_K f_\lambda(gk_2)\,dk_2\right)d(gK)$$

$$\operatorname{vol}(G/K)^{-1}\int_{(K/M)\times(a\cap\Omega)}\left(\int_K f_\lambda(\exp[\operatorname{Ad}(k)\,H]\,k_2)\,dk_2\right)$$

$$\times\left|\prod_{\alpha\in P_+}\sin\alpha(iH)\right| d(kM)\,dH.$$

(ii) We make the change of variables $H\to\lambda H$ in the formula (i), and use the facts that $V=\dim\mathbf{a}+\operatorname{card}(P_+)$, and that π_λ is one–one on $\lambda\Omega\times K$ to deduce that

$$\int_G f_\lambda(g)\,dg=\operatorname{vol}(G/K)^{-1}\lambda^{-\dim V}\int_{\mathbf{a}}\left|\prod_{\alpha\in P_+}\lambda\sin\lambda^{-1}\alpha(iH)\right|$$

$$\times\int_{K/M}\int_K f(\operatorname{Ad}(k)\,H,k_2)\,dk_2\,d(kM)\,dH.$$

Now there is a constant $C'>0$ such that

$$\left|\prod_{\alpha\in P_+}\lambda\sin\lambda^{-1}\alpha(iH)-\prod_{\alpha\in P_+}\alpha(iH)\right|\leqslant\frac{C'}{\lambda^2}$$

for all $\lambda>0$, uniformly with respect to $H\in\Omega$. Thus there is a constant $C>0$ such that

$$\left|\lambda^{\dim V}\int_G f_\lambda(g)\,dg-\operatorname{vol}(G/K)^{-1}\int_{\mathbf{a}}\left|\prod_{\alpha\in P_+}\alpha(iH)\right|\right.$$

$$\left.\times\int_{K/M}\int_K f(\operatorname{Ad}(k)\,H,k_2\,dk_2\,d(kM)\,dH\right|\leqslant\frac{C}{\lambda^2}\qquad(1)$$

for all sufficiently large λ. On the other hand, the Jacobian factor for the transformation β is [8, p. 382],

$$\beta^*(dv)=\left|\prod_{\alpha\in P_+}\alpha(iH)\right| d(kM)\,dH.$$

So

$$\int_{\mathfrak{a}} \left| \prod_{\alpha \in P_+} \alpha(iH) \right| \int_{K/M} \int_K f(\mathrm{Ad}(k)\, H, k_2)\, dk_2\, d(kM)\, dH$$

$$= \int_{V \rtimes K} f(v, k)\, dv\, dk, \tag{2}$$

and (ii) follows by combining (1) and (2).

4. Approximation of Fourier Transforms

The family of mappings π_λ give rise, as we have seen, to approximation between representations of $V \rtimes K$ and mapping $\mathscr{R}_\sigma \sigma \circ \pi_\lambda \mathscr{R}_\sigma^{-1}$ deriving from representations σ of G; moreover, there are associated approximations of integrals carried by two groups. The second phase in the study of the action of the mappings π_λ is to show that they give rise to approximations of traces of such a kind that one can effectively transfer Fourier analysis from G to $V \rtimes K$.

(4.1) Definitions. (i) If $f \in C_c(V \rtimes K)$ and $\sigma = \sigma_{\psi,\eta} \in \hat{G}$, write

$$\hat{f}_\lambda(\sigma) = \int_G f_\lambda(x)\, \sigma(x)\, dx.$$

This is a linear operator on $\mathscr{H}_\sigma$.

(ii) If $\sigma = \sigma_{\psi,\eta} \in \hat{G}$, write

$$\hat{f}(\mathscr{R}_\sigma \sigma \circ \pi_\lambda \mathscr{R}_\sigma^{-1}) = \int_{V \times K} f(x)\, \mathscr{R}_\sigma \sigma \circ \pi_\lambda(x)\, \mathscr{R}_\sigma^{-1}\, dx,$$

which is a bounded linear operator on $\mathscr{H}^\eta$ for all ψ.

A key ingredient in establishing the trace estimates is the fact that, if we begin with a fixed function f of the form $F \otimes h$ on $V \rtimes K$, where $F \in C_c(V)$ and h is a trigonometric polynomial on K, then the families of operators $\hat{f}_\lambda(\sigma)$ and $\hat{f}(\mathscr{R}_\sigma \sigma \circ \pi_\lambda \mathscr{R}_\sigma^{-1})$ (as λ and σ vary) are, in a certain sense, of "fixed finite rank": see Proposition (8.6). We shall need a chain of lemmas.

(4.2) Lemma. *Let h be a trigonometric polynomial on K. Then there exists a finite subset E of $\hat{M}$ such that*

$$\int_K h(k_0 k^{-1})\, u(k)\, dk = 0$$

for all $\eta \in \hat{M} \setminus E$, $u \in \mathscr{H}^\eta$ and all $k_0 \in K$.

Proof. If $u \in \mathscr{H}^\gamma$ then for each $k \in K$, $u(k) \in \mathscr{H}_\gamma$. Considering $\mathscr{H}_\gamma$ as a subrepresentation of $L^2(M)$ one easily checks that $u(k)(m) = u(km, e)$. Thus $u \to u(\cdot)(e)$ gives a mapping $\mathscr{H}^\gamma \to L^2(K)$ which intertwines the left regular actions of K. It is obvious that if $\gamma \neq \gamma'$ the images of $\mathscr{H}^\gamma$ and $\mathscr{H}^{\gamma'}$ are orthogonal.

To prove the lemma, suppose the contrary. Then since h is a trigonometric polynomial, the left regular representation of K on $\mathscr{H}^\eta$ would contain a certain fixed irreducible representation v as η runs through some infinite set of elements of $\hat{M}$. So it would follow that v occurs infinitely often in the left regular action of K on $L^2(K)$, which contradicts the Peter–Weyl theorem.

(4.3) Lemma. *Let h be a trigonometric polynomial on K, and let $e^{i\psi} \otimes \eta \in (MA)\hat{}$. There is a subspace $\mathscr{J}$ of $\mathscr{H}^\eta$ such that if $\mathscr{P}_\mathscr{J}$ denotes the orthogonal projection of $\mathscr{H}^\eta$ onto $\mathscr{J}$, then*

(i) $\mathscr{P}_\mathscr{J}(\mathscr{H}_{\psi+\gamma,\eta\,|\,K}) \subseteq \mathscr{H}_{\psi+\gamma,\eta\,|\,K}$ *for all* $\gamma \in \mathscr{P}$; *and*

(ii) $\int_K h(k_0 k^{-1})\, dk = 0$ *for all* $k_0 \in K$

and all $h \in \mathscr{J}^\perp$, the orthogonal complement of $\mathscr{J}$ in $\mathscr{H}^\eta$

Proof. As in (2.2) (i), fix $\gamma_0 \in \mathscr{P}$ such that γ_0 is generic, and consider the chain of K-invariant subspaces $\mathscr{H}_1 \subseteq \mathscr{H}_2 \subseteq \cdots \subseteq \mathscr{H}_n = \mathscr{H}_{\psi+n\gamma_0,\eta\,|\,K} \subseteq \cdots$. We claim that $\mathscr{J}$ can be taken to be $\mathscr{H}_{n_0}$, provided that n_0 is large enough.

First, observe that, if $\mathscr{P}_{n_0}$ denotes the projection mapping into $\mathscr{H}_{n_0}$, then (i) holds with $\mathscr{P}_\mathscr{J}$ replaced by $\mathscr{P}_{n_0}$. This is because the operator $\mathscr{P}_{n_0}$ is defined by convolution with an appropriate kernel P_{n_0} on K (by the Peter–Weyl theorem); the operation of convolution with the measure $P_{n_0}\, dk$ commutes with the restriction mapping (to K); and convolution with P_{n_0} leaves the space $\mathscr{H}_{\psi,\eta}(G)$ invariant.

If (ii) were not to hold for $\mathscr{J} = \mathscr{H}_{n_0}$, and n_0 sufficiently large, then the same irreducible representation of K would occur infinitely often in the decomposition of the left regular representation on $\mathscr{H}^\eta$, by (2.2)(iii). This is impossible.

We come now to the "finite rank" results.

(4.4) Proposition. *Let $f = F \otimes h$ be a function on $V \rtimes K$ such that $F \in C_c(V)$ and h is a trigonometric polynomial. Let E be the finite subset of $\hat{M}$ in* (4.2).

(i) *Suppose $\eta \in \hat{M} \setminus E$. Then*

$$\hat{f}(\rho_{\psi,\eta}) = 0 \qquad \text{for all } \psi \in \mathbf{a}^{*+}. \tag{1}$$

Suppose $\sigma = \sigma_{\psi,\eta}$ *with* $e^{i\psi} \otimes \eta \in (MA)\hat{}$ *and* $\lambda > 0$. *Then*

$$\hat{f}(\mathcal{R}_\sigma \sigma \circ \pi_\lambda \mathcal{R}_\sigma^{-1}) = 0 \tag{2}$$

and

$$\hat{f}_\lambda(\sigma) = 0. \tag{3}$$

(ii) *Suppose* $\eta \in E$ *and let* $\mathcal{J}$ *and* $\mathcal{P}_{\mathcal{J}}$ *be as in* (4.3). *Then*

$$\hat{f}(\rho_{\psi,\eta}) = \hat{f}(\rho_{\psi,\eta})\,\mathcal{P}_{\mathcal{J}} \tag{4}$$

for all $\psi \in a^{*+}$.

For all $e^{i\psi} \otimes \eta \in (MA)\hat{}$ *and for all* $\sigma = \sigma_{\psi+\gamma,\eta}$ *with* $\gamma \in \mathcal{P}$,

$$\hat{f}(\mathcal{R}_\sigma \sigma \circ \pi_\lambda \mathcal{R}_\sigma^{-1}) = \hat{f}(\mathcal{R}_\sigma \sigma \circ \pi_\lambda \mathcal{R}_\sigma^{-1})\,\mathcal{P}_{\mathcal{H}} \tag{5}$$

and

$$\mathcal{R}_\sigma \hat{f}_\lambda(\sigma)\,\mathcal{R}_\sigma^{-1} = \mathcal{R}_\sigma \hat{f}_\lambda(\sigma)\,\mathcal{R}_\sigma^{-1}\,\mathcal{P}_{\mathcal{J}}, \tag{6}$$

Statements (6) and (7) are λ-independent.

Proof. (i)(1) Suppose $u \in \mathcal{H}^\eta$ and that $\psi \in \mathbf{a}^{*+}$. Then

$$\begin{aligned}
[f(\rho_{(\psi,\eta)})](k_0) &= \int_{V \rtimes K} F(v)\, h(k) [\rho_{(\psi,\eta)}(v,k)\, u](k_0)\, dv\, dk \\
&= \int_V F(v)\, e^{-i\psi(\mathrm{Ad}(k_0^{-1})v)} \int_K h(h)\, u(k^{-1}k_0)\, dk\, dv \\
&= \int_V F(v)\, e^{i\psi(\mathrm{Ad}(k_0^{-1})v)} \int_K h(k_0 k^{-1})\, u(k)\, dk\, dv. \qquad (7)
\end{aligned}$$

If $\mu \in \hat{M} \backslash E$, the second integral in (7) is zero, by Lemma (4.2). So (1) follows.

(ii)(4) If $u \in \mathcal{H}^\eta$ we may write $u = \mathcal{P}_{\mathcal{J}} u + \mathcal{P}_{\mathcal{J}} \perp u$. Then applying (7) to $\mathcal{P}_{\mathcal{J}} u$ and $\mathcal{P}_{\mathcal{J}} \perp u$, and using Lemma (4.3), we establish (4).

(i)(2) If $u \in \mathcal{R}_\sigma \mathcal{H}_\sigma$, then

$$\begin{aligned}
&[\hat{f}(\mathcal{R}_\sigma \sigma \circ \pi_\lambda \mathcal{R}_\sigma^{-1})\, u](k_0) \\
&= \int_V \int_K F(v)\, h(k) [\mathcal{R}_\sigma^{-1} u] \left(\left(\exp \frac{v}{\lambda} k \right)^{-1} k_0 \right) dv\, dk \\
&= \int_V F(v)\, \sigma \left(\exp \frac{v}{\lambda} \right) \int_K h(k)\, [\mathcal{R}_\sigma^{-1} u](k^{-1} k_0)\, dk\, dv \\
&= \int_V F(v)\, \sigma \left(\exp \frac{v}{\lambda} \right) \int_K h(k_0 k^{-1}) [\mathcal{R}_\sigma^{-1} u](k)\, dk\, dv.
\end{aligned}$$

By Lemma (4.2) the second integral in (8) is zero if $\eta \notin E$.

(ii)(5) This follows from (8) and Lemma (4.3)

(i)(3) If $u \in \mathscr{H}_{\sigma|K}$ and $k_0 \in E$, then by (3.3)

$$
\begin{aligned}
[\hat{f}_\lambda(\sigma)\,\mathscr{R}_\sigma^{-1}u](k_0) &= \int_G f_\lambda(g)[\mathscr{R}_\sigma^{-1}u](gk_0)\,dg \\
&= \mathrm{vol}(G/K)^{-1} \int_{\mathbf{a}} \Big|\prod_{\alpha \in P_+} \sin \alpha(iH)\Big| \\
&\quad \times \int_{K/M} \int_K f_\lambda(\exp(\mathrm{Ad}(k_1)\,H)\,k_2) \\
&\quad \times [\mathscr{R}_\sigma^{-1}u](\exp(\mathrm{Ad}(k_1)\,H)\,k_2k_0)\,dk_2\,d(k_1M)\,dH \\
&= \mathrm{vol}(G/K)^{-1} \int_{\mathbf{a}} \Big|\prod_{\alpha \in P_+} \sin \alpha(iH)\Big| \int_{K/M} F(\lambda\,\mathrm{Ad}(k_1)\,H) \\
&\quad \times \int_K [\sigma(\exp(-\mathrm{Ad}(k_1)\,H))\,\mathscr{R}_\sigma^{-1}u](k_2)\,h(k_2k_0^{-1}) \\
&\quad \times dk_2\,d(k_1M)\,dH. \qquad (9)
\end{aligned}
$$

By Lemma (4.2), the last integral is zero if $\eta \in \hat{M}\backslash E$.

(ii)(6) This follows from formula (9) and Lemma (4.3).

Theorem (2.2) and Lemma (4.4) will now be combined to produce operator norm approximations for Fourier transforms (4.5), and subsequently trace approximations ((4.6), (4.7))

(4.5) PROPOSITION. *The notation and assumption are as in* (4.4). *Let M be a fixed positive number. Then there exists a constant C, determined by f and M, such that*

$$\|\hat{f}(\rho_{(\psi+\gamma)/\lambda,\eta}) - \mathrm{vol}(G/K)\,\lambda^{\dim V}\mathscr{R}_\sigma \hat{f}_\lambda(g)\,\mathscr{R}_\sigma^{-1}\|_{\mathbf{B}(\mathscr{H}^\eta)} \leqslant \frac{C}{\lambda} \qquad (10)$$

for all sufficiently large λ, all $\eta \in \hat{M}$, and all $\gamma \in \mathscr{P}$ such that $\|\gamma/\lambda\| \leqslant M$, where $\sigma = \sigma_{(\psi+\gamma),\eta}$.

Proof. The left side of (10) is zero for all but finitely many η ((4.2)(1)). So it will suffice to prove (10) for a fixed $\eta \in \hat{M}$. By (4.4)(ii), the left side of (10) is the same as

$$\sup_{u \in \mathscr{I}, \|u\| \leqslant 1} \|\hat{f}(\rho_{(\psi+\gamma)/\lambda,\eta})\,u - \mathrm{vol}(G/K)\,\lambda^{\dim V}\mathscr{P}_\sigma \hat{f}_\lambda(\sigma)\,\mathscr{R}_\sigma^{-1}u\|_{\mathscr{H}^\eta}. \qquad (11)$$

Since the space $\mathscr{J}$ is finite dimensional, it will be enough to estimate the norm difference in (11) for a fixed $u \in \mathscr{J}$.

The approximation theorem (2.2) implies the existence of a constant C_1 such that for $\sigma = \sigma_{(\psi+\gamma),\eta}$

$$\|\hat{f}(\rho_{(\psi+\gamma)/\lambda,\eta})\,u - \hat{f}(\mathscr{R}_\sigma \sigma \circ \pi_\lambda \mathscr{R}_\sigma^{-1})\,u\| \leqslant \frac{C_1}{\lambda} \tag{12}$$

for all $\psi \in \mathscr{P}$ such that $\|\gamma/\lambda\| \leqslant M$. But

$$\hat{f}(\mathscr{R}_\sigma \sigma \circ \pi_\lambda \mathscr{R}_\sigma^{-1})(u) = \mathscr{R}_\sigma \int_{V \times K} f(v, k)\,\sigma\pi_\lambda(v, k)\,dv\,dk\,\mathscr{R}_\sigma^{-1}u.$$

According to (3.3),

$$\begin{aligned}
&\operatorname{vol}(G/K)\,\lambda^{\dim V} \int_G f_\lambda(g)\,\sigma(g)\,dg \\
&\quad= \lambda^{\dim V} \int_{\mathfrak{a}} \left|\prod_{\alpha \in P_+} \sin \alpha(iH)\right| \int_{K/M} \int_K f_\lambda(\exp(\operatorname{Ad}(k)\,H)\,k_2) \\
&\qquad\times \sigma(\exp(\operatorname{Ad}(k)\,H)\,k_2)\,d(kM)\,dH \\
&\quad= \int_{\mathfrak{a}} \left|\prod_{\alpha \in P_+} \lambda \sin \alpha(iH)/\lambda)\right| \int_{K/M} \int_K f_\lambda(\exp(\operatorname{Ad}(k)\,H/\lambda)\,k_2) \\
&\qquad\times \sigma(\exp(\operatorname{Ad}(k)\,H/\lambda)\,k_2\,dk_2\,d(kM)\,dH \\
&\quad= \int_{\mathfrak{a}} \left|\prod_{\alpha \in P_+} \cdots\right| \int_{K/M} \int_K f(\operatorname{Ad}(k)\,H, k_2)\,\sigma \circ \pi_\lambda(\operatorname{Ad}(k)\,H, k_2)\,dk_2\,d(kM)\,dH \\
&\quad= \int_{V \times K} f(v, k)\,\sigma \circ \pi_\lambda(v, k)\,dv\,dk \\
&\qquad\times \int_{\mathfrak{a}} \left|\prod_{\alpha \in P_+} \alpha(iH)\right| \int_{K/M} \int_K W(\lambda, H)\,f(\operatorname{Ad}(k)\,H, k_2) \\
&\qquad\times \sigma \circ \pi_\lambda(\operatorname{Ad}(k)\,H, k_2)\,dk_2\,d(kM)\,dH,
\end{aligned} \tag{13}$$

where

$$W(\lambda, H) = \left|\prod_{\alpha \in P_+} \frac{\sin \alpha(iH/\lambda)}{\alpha(iH/\lambda)}\right| - 1.$$

Now $W(\lambda, H)$ may be regarded as an $\operatorname{Ad}(K)$-invariant function on V; and

$$|W(\lambda, v)| = O\left(\frac{1}{\lambda^2}\right) \tag{14}$$

as $\lambda \to +\infty$ uniformly with respect to $v \in B$ (notation as in (3.2)). Furthermore, we may appeal to the proof of Lemma (3.3)(ii) to write the second term on the right side of (13) as

$$\int_{V \times K} f(v, k)\, W(\lambda, v)\, \sigma \circ \pi_\lambda(v, k)\, dv\, dk. \tag{15}$$

We deduce from (13) and (15) that

$$\hat{f}(\mathscr{R}_\sigma \sigma \circ \pi_\lambda \mathscr{R}_\sigma^{-1}) = \mathrm{vol}(G/K)\, \lambda^{\dim V} \hat{f}_\lambda(\mathscr{R}_\sigma \sigma \circ \pi_\lambda \mathscr{R}_\sigma^{-1})$$

$$\int_{V \times K} f(v, k)\, W(\lambda, v)\, \mathscr{R}_\sigma \sigma \circ \pi_\lambda(v, k)\, \mathscr{R}_\sigma^{-1}\, dv\, dk. \tag{16}$$

But we may also write the last term in (16) as

$$\int_{V \times K} f(v, k)\, W(\lambda, v)\, \rho_{(\psi + \gamma)/\lambda, \eta}(v, k)\, dv\, dk + O\left(\frac{1}{\lambda}\right) \tag{17}$$

as $\lambda \to +\infty$ by Theorem 2.2 and the fact that W is uniformly bounded. The $O(1/\lambda)$ term in (17) has a bound depending only on f and M. Yet the integral in (17) is bounded in absolute value by $\|f\|_1\, \|W(\lambda, .)\|_\infty$, which by (14) is $O(1/\lambda^2)$. By combining these last observation with (12), we see that the proof is complete.

(4.6) Lemma. *Let Φ be a bounded operator-valued function on X, and f a function on $V \rtimes K$ of the form $f = F \otimes h$, where $F \in C_c(V)$ and h is a trigonometric polynomial on K. Let M be a fixed positive number, and $\Omega' \times K$ a given compact subset of $V \rtimes K$. Then there exists a constant C (depending only on f and M) such that*

$$\begin{aligned} &|\mathrm{Tr}\{(\Phi \hat{f})(\rho_{(\psi + \gamma)/\lambda, \eta})\, \rho_{(\psi + \gamma/\lambda, \eta)}(v, k)\} \\ &\quad - \lambda^{\dim V}\, \mathrm{vol}(G/K)\, \mathrm{Tr}\{(\Phi^{(\lambda)} \hat{f}_\lambda)(\sigma_{\psi + \gamma, \eta})\, \sigma_{\psi + \gamma, \eta} \circ \pi_\lambda(v, k)\}| \leqslant \frac{C}{\lambda} \end{aligned} \tag{18}$$

for all $e^{i\psi} \otimes \eta \in (MA)^\wedge$, $\gamma \in \mathscr{P}$ such that $\|(\psi + \gamma)/\lambda\| \leqslant M$, all $(v, k) \in \Omega' \times K$, and all sufficiently large λ. (The other notation is as in (4.4).)

Proof. We shall use the inequality

$$|\mathrm{Tr}(UV)| \leqslant U\|_{\beta(E)}\, \|V\|_{\psi_1}$$

for linear operators on the finite-dimensional Hilbert space E (cf. [9, D. 39]).

For simplicity of presentation, we shall drop the subscripts on ρ. Then if $(v, k) \in V \times K$, and if $\sigma = \sigma_{\psi + \gamma\eta}$,

$$\begin{aligned} &|\mathrm{Tr}\{(\Phi\hat{f})(\rho)\,\rho(v,k)\} - \lambda^{\dim V}\mathrm{vol}(G/K)\,\mathrm{Tr}\{(\Phi^{(\lambda)}\hat{f}_\lambda)(\sigma)\,\sigma \circ \pi_\lambda(v,k)\}\}| \\ &\quad = |\mathrm{Tr}\{\Phi(\rho)[\hat{f}(\rho)\,\rho(v,k) \\ &\qquad - \lambda^{\dim V}\mathrm{vol}(G/K)\,\mathscr{R}_\sigma \hat{f}_\lambda(\sigma)\,\mathscr{R}_\sigma^{-1}\,\mathscr{P}_{\mathscr{I}}\,\mathscr{R}_\sigma\,\sigma(\pi_\lambda(v,k))\,\mathscr{R}_\sigma^{-1}\mathscr{P}_\sigma]| \quad (19) \end{aligned}$$

by (4.4)(6) and the symmetry of the trace. But the right side of (19) is at most

$$\begin{aligned} &\|\Phi(\rho)\|_{(\mathscr{H}^\eta)}\,\|[\cdots]\|_{\psi_1} \\ &\quad \leqslant \|\Phi(\rho)\|\,\{\|[\hat{f}(\rho) - \lambda^{\dim V}\mathrm{vol}(G/K)\,\mathscr{R}_\sigma \hat{f}_\lambda(\sigma)\,\mathscr{R}_\sigma^{-1}]\,\rho(v,k)\|_{\psi_1} \\ &\qquad + \lambda^{\dim V}\mathrm{vol}(G/K)\|\mathscr{R}_\sigma \hat{f}_\lambda(\sigma)\,\mathscr{R}_\sigma^{-1}[\mathscr{P}_{\mathscr{I}}\,\rho(v,k) \\ &\qquad - \mathscr{P}_{\mathscr{I}}\,\mathscr{R}_\sigma\,\sigma(\pi_\lambda(v,k))\,\mathscr{R}_\sigma^{-1}]\|_{\psi_1}\}. \end{aligned} \quad (20)$$

By (8.6)(4). Since ρ is unitary [9, D. 39], implies that the first term inside the curly brackets is at most

$$\begin{aligned} &\|[\hat{f}(\rho) - \lambda^{\dim V}\mathrm{vol}(G/K)\,\mathscr{R}_\sigma \hat{f}_\lambda(\sigma)\,\mathscr{R}_\sigma^{-1}]\,\mathscr{P}_{\mathscr{I}}\|_{\psi_1} \\ &\quad \leqslant d_0\|\hat{f}(\rho) - \lambda^{\dim V}\mathrm{vol}(G/K)\,\mathscr{R}_\sigma \hat{f}_\lambda(\sigma)\,\mathscr{R}_\sigma^{-1}\|_{\mathscr{B}(\mathscr{H}^\eta)} \\ &\quad \leqslant \frac{C_1}{\lambda} \end{aligned} \quad (21)$$

for all sufficiently large λ, by (4.5). The first inequality in (21) results from the fact that $\mathscr{P}_{\mathscr{I}}$ projects $\mathscr{H}^\eta$ onto a finite-dimensional space $\mathscr{I}$ of dimension say d_0.

Similarly, the second term inside the curly brackets in (20) is at most

$$\lambda^{\dim V}\mathrm{vol}(G/K)\|\mathscr{R}_\sigma \hat{f}_\lambda(\sigma)\,\mathscr{R}_\sigma^{-1}\|_{\mathscr{B}(\mathscr{H}^\eta)}\,\|\mathscr{P}_{\mathscr{I}}[\rho(v,k) - \mathscr{R}_\sigma\,\sigma(\pi_\lambda(v,k))\,\mathscr{R}_\sigma^{-1}\mathscr{P}_\sigma]\|_{\phi_1}. \quad (22)$$

By (4.5), there is a constant C_2 such that

$$\begin{aligned} &\|\lambda^{\dim V}\mathrm{vol}(G/K)\,\mathscr{R}_\sigma \hat{f}_\lambda(\sigma)\,\mathscr{R}_\sigma^{-1}\|_{\mathscr{B}(\mathscr{H}^\eta)} \\ &\quad \leqslant \|\hat{f}(\rho)\|_{\mathscr{B}(\mathscr{H}^\eta)} + \frac{C_2}{\lambda} \\ &\quad \leqslant \|f\|_1 + \frac{C_2}{\lambda} \end{aligned} \quad (23)$$

for all sufficiently large λ. Also

$$\|\mathscr{P}_{\gamma}[\rho(v,k)-\mathscr{R}_{\sigma}\sigma(\pi_{\lambda}(v,k)\,\mathscr{R}_{\sigma}^{-1}\mathscr{P}_{\sigma}]\|_{\psi_1}$$
$$\leqslant d_0\|\rho(v,k)-\mathscr{R}_{\sigma}\sigma(\pi_{\lambda}(v,k))\,\mathscr{R}_{\sigma}^{-1}\mathscr{P}_{\sigma}\|_{\mathscr{B}(\mathscr{H}^{\eta})}$$
$$\leqslant \frac{C_3}{\lambda} \tag{24}$$

for all $(v,k)\in\Omega'\times K$, all $\gamma\in\mathscr{P}$ such that $\|\gamma/\lambda\|\leqslant M$, and all of the finitely many $\eta\in\hat{M}$ for which the left side of (18) is not identically zero. This is due to Theorem (2.2).

The inequality (19) results by collecting together the inequalities (21)–(25).

We come now to the trace estimate that lies at the heart of the transfer of harmonic analysis from G to $V\rtimes K$.

(4.7) PROPOSITION. *Let f and g be functions on $V\rtimes K$ having the same form as the function f of* (4.6). *The other notation is as in* (4.6) *also. Then there exists a constant C depending on f, g, and M, such that*

$$|\mathrm{Tr}\{\Phi(\rho)\,\hat{f}(\rho)\,\hat{g}(\rho)\}$$
$$-\lambda^{2\dim V}\mathrm{vol}(G/K)^2\,\mathrm{Tr}\{\Phi^{(\lambda)}(\sigma)\,\hat{f}_{\lambda}(\sigma)\,\hat{g}_{\lambda}(\sigma)\}|\leqslant\frac{C}{\lambda}$$

for all $\eta\in\hat{m}$, all $\gamma\in\mathscr{P}$ such that $\|\gamma/\lambda\|\leqslant M$, and all sufficiently large $\lambda>0$.

(The symbol ρ stands for $\rho_{(\psi+\gamma)/\lambda,\eta}$ and the symbol σ for $\sigma_{\psi+\gamma,\eta}$ as in the proof of (4.6).)

Proof. The function g has compact support. So we may multiply the inequality (19) by $g(v,k)$ and integrate over $V\rtimes K$ to see that

$$|\mathrm{Tr}\{\Phi(\rho)\,\hat{f}(\rho)\,\hat{g}(\rho)\}$$
$$-\lambda^{\dim V}\mathrm{vol}(G/K)\,\mathrm{Tr}\{\Phi^{(\lambda)}(\sigma)\,\hat{f}_{\lambda}(\sigma)\,\hat{g}(\sigma\circ\pi_{\lambda})\}|\leqslant\frac{C_1}{\lambda} \tag{25}$$

say, under the stated conditions of Lemma (4.6). But

$$\|\hat{g}(\sigma\circ\pi_{\lambda})-\lambda^{\dim V}\mathrm{vol}(G/K)\,\hat{g}_{\lambda}\mathscr{R}_{\sigma}\sigma\mathscr{R}_{\sigma}^{-1})\|_{\mathscr{B}(\mathscr{H}^{\eta})}\leqslant\frac{C_2}{\lambda} \tag{26}$$

for some constant C_2 and all sufficiently large λ such that $\|\gamma/\lambda\|\leqslant M$ by (16), (17). We now imitate the arguments in the proof of Lemma (4.6) (cf. especially (21) to show that the term $\hat{g}(\sigma\circ\pi_{\lambda})$ in (25) can be replaced by

$$\lambda^{\dim V}\mathrm{vol}(G/K)\,\hat{g}_{\lambda}(\mathscr{R}_{\sigma}\sigma\mathscr{R}_{\sigma}^{-1})$$

while committing an error that is uniformly $O(1/\lambda)$. One needs (26) and the fact that

$$\begin{aligned}
&\|\lambda^{\dim V}\mathrm{vol}(G/K)\,\mathcal{R}_\sigma \hat{f}_\lambda(\sigma)\,\mathcal{R}_\sigma^{-1}\|_{\psi_1}\\
&\quad=\|\lambda^{\dim V}\mathrm{vol}(G/K)\,\mathcal{R}_\sigma \hat{f}_\lambda(\sigma)\,\mathcal{R}_\sigma^{-1}\,\mathcal{P}_\gamma\|_{\psi_1}\\
&\quad\leqslant d_0\|\lambda^{\dim V}\mathrm{vol}(G/K)\,\mathcal{R}_\sigma \hat{f}_\lambda(\sigma)\,\mathcal{R}_\sigma^{-1}\|_{\mathcal{B}(\mathcal{H}^\gamma)}\\
&\quad\leqslant d_0\left(\|f\|_1+\frac{C_3}{\lambda}\right)
\end{aligned}$$

for all sufficiently large λ, by (24). We omit the details.

5. Proof of Theorem (2.2)

To be able to perform the final estimates that make up the proof of Theorem (2.2), it will be necessary to introduce a regularisation procedure. We shall employ the results of J.-L. Clerc concerning summability with respect to a certain family of Riesz–Bochner means.

(5.1)

According to [2], it is possible to choose a metric, a bi-invariant Laplacian on G, an inner product κ on $\mathbf{t}^{\mathbb{C}}$ and a dual inner product also denoted κ, such that

(a) κ agrees with the Killing form on $\mathbf{t}'^{\mathbb{C}}$;

(b) $\mathbf{z}$ is orthogonal to $\mathbf{t}'^{\mathbb{C}}$ with respect to κ;

(c) each $\chi\in\hat{T}^+$ is an eigenfunction of the Laplacian, and the eigenvalue $\mathscr{E}(\chi)$ corresponding to χ is

$$\mathscr{E}(\chi)=-\kappa(d\chi+\delta,\ d\chi+\delta)+\kappa(\delta,\ \delta) \tag{1}$$

δ denoting one-half of the sum of the positive roots.

We shall abbreviate $\kappa(\cdot,\cdot)$ to $(\cdot,\cdot)$ in what follows. For each $R>0$, one defines the function S_R on $\hat{G}$ by the formula

$$S_R(\sigma_\chi)=\left(1-\frac{\mathscr{E}(\chi)}{R^2}\right)_+^n. \tag{2}$$

In (2), n denotes the dimension of G; and, if α is real,

$$\alpha_+=\begin{cases}\alpha & \text{if}\quad \alpha\geqslant 0\\ 0 & \text{otherwise.}\end{cases}$$

It follows from [2], Théorème 3 that the Fourier multipliers.

$$\{S_R(\sigma_\chi)\, I_\chi\}_{\chi \in \hat{T}^+} \qquad (R > 0)$$

are uniformly bounded on $L^p(G)$ whenever $1 \leqslant p < \infty$. (The operator I_χ is the identity operator on $\mathscr{H}_\chi$.) Moreover,

$$\left\| f - \sum_\chi d_\chi S_R(\sigma_\chi) \operatorname{Tr}\{\hat{f}(\sigma_\chi)\, \sigma_\chi\} \right\|_p \to 0$$

as $R \to \infty$.

(5.2)

We associate with the family $\{S_R\}_{R>0}$ two families of functions, carried by X and by a subset of X, respectively.

DEFINITION. Suppose that $R > 0$.

(i) Define the function Ψ_R on X by writing

$$\Psi_R(\rho_{\psi,\eta}) = \left(1 - \frac{(\psi, \psi)}{R^2}\right)_+^n. \tag{3}$$

In (3), $\eta \in \hat{M}$ and $\psi \in \mathbf{a}^{*+}$.

(ii) For $e^{i\psi} \otimes \eta \in (MA)\hat{}$, $\gamma \in \mathscr{P}$, define

$$\mathscr{S}_R(\rho_{\psi+\gamma,\eta}) = 1 - \frac{(\psi + \gamma + \delta, \psi + \gamma + \delta) - (\delta, \delta)}{R^2}. \tag{4}$$

Notation. If $\eta \in \hat{M}$, we write d_η for the dimension of η, and we write $d_{\psi+\gamma,\eta}$ for the dimension of $\sigma_{\psi+\gamma,\eta}$.

We proceed to establish relationships between d_η and $d_{\psi+\gamma,\eta}$ that will be crucial in the estimates to follow.

DEFINITION. We say that Q is a *proper sub-cone* of $\mathbf{a}^{*+}$ if

(i) $Q \in \mathbf{a}^{*+}$, and Q is a cone;

(ii) for every sequence $\{\phi\}$ in Q that tends to infinity, we have $\phi(H_\alpha) \to +\infty$ for every $\alpha \in P_+$.

Observe that the cone $\mathbf{a}^{*+}$ can be exhausted by an increasing sequence of proper sub-cones.

(5.3) LEMMA. *Suppose that* $e^{i\psi} \otimes \eta \in (MA)^\wedge$

(i) *There is a constant* $C > 0$ *such that*

$$\left| d_{\psi+\gamma,\eta} \prod_{\alpha \in P_+} (\delta, \alpha) - d_\eta \prod_{\alpha \in P_+} (\psi + \gamma, \alpha) \right| \leqslant C\, d_\eta \prod_{\alpha \in P_-} (\psi + \gamma, \alpha)$$

for all $\gamma \in \mathscr{P}$.

(ii) *Given any proper sub-cone* Q *of* $\mathbf{a}^{*+}$, *and* $\varepsilon > 0$, *there exists a constant* D *such that*

$$\left| d_{\psi+\gamma,\eta} \prod_{\alpha \in P_-} (\delta, \alpha) - d_\eta \prod_{\alpha \in P_+} (\psi + \gamma, \alpha) \right| \leqslant \varepsilon\, d_\eta \prod_{\alpha \in P_-} (\psi + \gamma, \alpha)$$

for all $\gamma \in \mathscr{P}$ *such that* $\|\gamma\| \geqslant D$ *and* $\gamma \in Q$.

Proof. (i) This is a simple consequence of the Weyl dimension formula. On the other hand, as $d\gamma \to \infty$ in Q, the same product (9) tends to $+\infty$. So the second product on the right side of (8) tends to 1, whence (ii).

(5.5) LEMMA. *Suppose that* $\eta \otimes e^{i\psi} \in (MA)^\wedge$ *and* $R > 0$. *Then there is a constant* $C > 0$ *such that, for all* $\lambda > 0$,

$$\sum_{\|\gamma\| \leqslant \lambda R} d_{\psi+\gamma,\eta} \leqslant C(\lambda R)^{\dim V}.$$

Proof. It follows from (7) that

$$\begin{aligned} d_{\psi+\gamma,\eta} &\leqslant C_1 \prod_{\alpha \in P_+} \|\psi + \gamma + \delta\| \\ &\leqslant C_2\, \|\gamma\|^{\operatorname{card}(P_+)} \end{aligned}$$

for certain constants C_1 and C_2. So

$$\begin{aligned} \sum_{\|\gamma\| \leqslant \lambda R} d_{\psi+\gamma,\eta} &\leqslant C_2 \sum_{\|\gamma\| \leqslant \lambda R} \|\gamma\|^{\operatorname{card}(P_+)} \\ &\leqslant C_3 \int_{|\phi| \leqslant \lambda R} |\phi|^{\operatorname{card}(P_+)}\, d\phi \\ &= C_4 \int_0^{\lambda R} r^{\dim(a) + \operatorname{card}(P_+) - 1}\, dr \\ &= C_5 (\lambda R)^{\dim V}. \end{aligned}$$

We come now to the final steps in the proof of the theorem

(5.6) LEMMA. (a) *To prove Theorem* (2.2), *it will suffice to establish the existence of a constant B such that*

$$\left| \sum_{\eta \in \hat{M}} d_\eta \int_{a^{*+}} \Psi_R(\rho_{\psi,\eta}) \operatorname{Tr}\{\Phi(\rho_{\psi,\eta}) \hat{f}(\rho_{\psi,\eta}) \hat{g}(\rho_{\psi,\eta})\} \left| \prod_{\alpha \in P_+} \phi(H_\alpha) \right| d\phi \right| \leqslant B\|f\|_p \|g\|_{p'}, \tag{10}$$

for all functions f and g in the tensor product of $C_c(V)$ *with the space of trigonometric polynomials on K, and all* $R > 0$.

(b) *For each* $\eta \in \hat{M}$, *make a fixed (but arbitrary) choice of some* $\psi = \psi(\eta) \in \mathbf{a}^*$ *such that* $\eta \otimes e^{i\psi} \in (MA)\hat{}$. *The left-hand side of* (10) *is a scalar multiple of*

$$\lim_{\lambda \to +\infty} \left| \sum_{\eta \in \hat{M}} \sum_{\gamma \in \mathscr{P}} d_{\psi+\gamma,\eta} S_{\lambda R}(\rho_{(\psi+\gamma)/\lambda,\eta}) \operatorname{Tr}\{\Phi(\rho_{(\psi+\gamma)/\lambda,\eta} \hat{f}(\rho_{(\psi+\gamma)/\lambda,\eta} \hat{g}(\rho_{\psi+\gamma/\lambda,\eta})\} \right. \tag{11}$$

Proof. (a) follows from the dominated convergence theorem and the fact that the element of Plancherel measure on X is

$$d(\rho_{\psi,\eta}) = d_\eta \left| \prod_{\alpha \in P_+} \phi(H_\alpha) \right| d\phi.$$

See [11].

(b) It will be helpful to recall that, for a fixed pair of functions f, g of the form specified the sum in (5) over $\hat{M}$ is actually a sum over a finite set E (Proposition (4.41). It is now easy to see that, by the definition of Riemann integral, the left side of (10) is the same as a multiple of

$$\lim_{\lambda \to +\infty} \left| \sum_{\eta \in \hat{M}} d_\eta \lambda^{-\dim V} \sum_{\gamma \in \hat{M}} \Psi_R(\rho_{(\psi+\gamma)/\lambda,\eta}) \operatorname{Tr}\{\Phi(\rho) \hat{f}(\rho) \hat{g}(\rho)\}, \left| \prod_{\alpha \in P_+} (\psi+\gamma)(H_\alpha) \right| . \right. \tag{12}$$

In (12), we have dropped the subscripts on all but one of the ρ's, and have written ψ for $\psi(\eta)$.

There are now two steps to make. (i) The first is to show that one can replace $\Psi_R(\rho_{(\psi+\gamma)/\lambda,\eta})$ by $S_{\lambda R}(\rho_{(\psi+\gamma)/\lambda,\eta})$. (ii) The second is to show that one can replace

$$d_\eta \prod_{\alpha \in P_+} (\psi+\gamma)(H_\alpha)$$

by $d_{\psi+\gamma,\eta}$.

As to (i), it is simple to see that there is a constant $C>0$ such that

$$S_{\lambda R}(\rho_{\psi+\gamma/\lambda,\eta}) = \Psi_R(\rho_{\psi+\gamma/\lambda,\eta}) = 0 \tag{13}$$

if $\|\gamma\| > C\lambda R$, and $\eta \in E$. Moreover,

$$|\Psi_R(\rho_{\psi+\gamma/\lambda,\eta}) - S_{\lambda R}(\rho_{\psi+\gamma/\lambda,\eta})| = O\left(\frac{1}{\lambda}\right) \tag{14}$$

as $\lambda \to +\infty$, for each $\eta \in \hat{M}$, uniformly with respect to $\gamma \in \mathscr{P}$ Both (13) and (14) follow from a careful examination of the definitions of (3) and (4).

Granted (13) and (14), we see that

$$\varlimsup_{\lambda \to +\infty} \Bigg| \sum_{\eta \in \hat{M}} d_\eta \lambda^{-\dim V} \sum_{\gamma \in \mathscr{P}} \{\Psi_R(\rho_{(\psi+\gamma)/\lambda,\eta}) - S_{\lambda R}(\rho_{(\psi+\gamma)/\lambda,\eta})\}$$

$$\mathrm{Tr}\{\Phi(\rho)\hat{f}(\rho)\,\hat{g}(\rho)\} \Bigg| \prod_{\alpha \in P_+} (\psi+\gamma)(H_\alpha) \Bigg| \Bigg|$$

$$\leqslant \varlimsup_{\lambda \to +\infty} O\left(\frac{1}{\lambda}\right) \sum_{\eta \in E} d_\eta \sum_{\|\gamma\| \leqslant CR\lambda} |\mathrm{Tr}\{\cdots\}| \left| \prod_{\alpha \in P_+} \frac{(\psi+\gamma)(H_\alpha)}{\lambda} \right| \lambda^{-\dim(a)}. \tag{15}$$

But as $\lambda \to +\infty$, the sum in (15) approaches a multiple of

$$\int_{|\phi| \leqslant CR} |\mathrm{Tr}\{\Phi(\rho_{\phi,\eta})\,\hat{f}(\rho_{\phi,\eta}\ \hat{g}(\rho_{\phi,\eta})\}| \prod_{\alpha \in P_+} \phi(H_\alpha)\, d\phi$$

which is finite. So the limit in (15) is zero, and (i) is established.

As to (ii), we wish to compare

$$I_1^{(\lambda)} = \sum_{\eta \in E} d_\eta \lambda^{-\dim V} \sum_{\gamma \in \hat{M}} S_{\lambda R}(\rho_{\psi+\gamma/\lambda,\eta)})\,\mathrm{Tr}\{\cdots\} \prod_{\alpha \in P_+} (\psi+\gamma),\alpha) \tag{16}$$

with

$$I_2^{(\lambda)} = \sum_{\eta \in E} \lambda^{-\dim V} \sum_{\gamma \in \mathscr{R}} d_\eta\ {}_{\lambda R}(\rho_{(\psi+\gamma)/\lambda,\eta})\,\mathrm{Tr}\{\cdots\}. \tag{17}$$

Let $\varepsilon > 0$ be given. Choose a proper subcone Q of a^{*+} such that

$$\sum_{\eta \in E} d_\eta \int_{\phi \in a^{*+} \setminus Q} |\mathrm{Tr}\{\Phi(\rho_{\phi,\eta})\,\hat{f}(\rho)\hat{g}(\rho)\}| \prod_{\alpha \in P_+} \phi(H_\alpha)\, d\phi < \varepsilon. \tag{18}$$

Let $I_{1,Q}^{(\lambda)}$ and $I_{2,Q}^{(\lambda)}$ denote the partial sums of (16) and (17), respectively, in which the second summation extends over $\gamma \in Q$. Similarly, denote by $I_{1,Q}^{(\lambda)}$ and $I_{2,Q}^{(\lambda)}$, the sums determined by the complementary set $Q' = a^{*+} \setminus Q$.

By Lemma (5.4)(ii) and (13),

$$\varlimsup_{\lambda \to +\infty} \left| I_{2,Q}^{(\lambda)} \prod_{\alpha \in P_+} (\delta, \alpha) - I_{1,Q}^{(\lambda)} \right|$$
$$\leqslant \varepsilon \varlimsup_{\lambda \to +\infty} \sum_{\eta \in E} d_\eta \lambda^{-\dim V} \sum_{\|\gamma\| \leqslant \mathscr{C} \lambda R} |\mathrm{Tr}\{\Phi(\rho_{(\psi+\gamma)/\lambda,\eta})\, \hat{f}(\rho)\, \hat{g}(\rho)\}|$$
$$\times \prod_{\alpha \in P_+} (\psi + \gamma, \alpha)$$
$$\leqslant \varepsilon \int_{\|\phi\| \leqslant CR} |\mathrm{Tr}\{\Phi(\rho)\, \hat{f}(\rho)\, \hat{g}(\rho)\}|\, d\rho. \tag{19}$$

In estimating the other difference, we use Lemma (5.5). We see that

$$\varlimsup_{\lambda \to +\infty} \left| I_{2,Q}^{(\lambda)} \prod_{\alpha \in P_+} (\delta, \alpha) - I_{2,Q}^{(\lambda)} \right|$$
$$\leqslant C \varlimsup_{\lambda \to +\infty} \prod_{\eta \in E} d_\eta \lambda^{-\dim V} \sum_{\gamma \in Q'} |\mathrm{Tr}\{\Phi(\rho_{(\psi+\gamma)/\lambda,\eta})\, \hat{f}(\rho)\, \hat{g}(\rho)\}|$$
$$\times \prod_{\alpha \in P_+} (\psi + \gamma, \alpha)$$
$$\leqslant C_2 \int_{\substack{\phi \in Q' \\ \eta \in E}} |\mathrm{Tr}\{\Phi(\rho_{\phi,\eta})\, \hat{f}(\rho)\, \hat{g}(\rho)\}|\, d\rho \leqslant \varepsilon C_2$$

by (18). The inequalities (19) and (20) establish (ii).

(5.7) Lemma. *Suppose $\lambda > 0$. Define*

$$I_3^{(\lambda)} = \sum_{\eta \in E} \sum_{\gamma \in \mathscr{P}} \mathrm{vol}(G/K)^2\, \lambda^{\dim V}\, d_{\psi+\gamma,\eta}\, S_{\lambda R}(\sigma_{\psi+\gamma,\eta})$$
$$\times \mathrm{Tr}\{\Phi^{(\lambda)}(\sigma)\, \hat{f}_\lambda(\sigma)\, \hat{g}_\lambda(\sigma)\}. \tag{21}$$

Then $\varlimsup_{\lambda \to +\infty} I_2^{(\lambda)} - I_3^{(\lambda)} = 0$. ($I_2^{(\lambda)}$ is defined by (17).)

Proof. By Lemma (8.9) there is a constant C_1 such that

$$|\mathrm{Tr}\{\Phi(\rho_{(\psi+\gamma)/\lambda,\eta})\, \hat{f}(\rho)\, \hat{g}(\rho) - \mathrm{vol}(G/K)^2\, \lambda^{2\dim V}$$
$$\times \mathrm{Tr}\{\Phi^{(\lambda)}(\sigma_{\psi+\gamma,\eta})\, \hat{f}_\lambda(\sigma)\, \hat{g}_\lambda(\sigma)\}|$$
$$\leqslant \frac{C_1}{\lambda} \tag{22}$$

for all $\eta \in \hat{M}$, all $\gamma \in \mathscr{P}$ and all sufficiently large λ. By (22), (13), and Lemma (9.5),

$$\begin{aligned}\overline{\lim_{\lambda \to +\infty}} |I_2^{(\lambda)} - I_3^{(\lambda)}| &\leqslant C_2 \overline{\lim_{\lambda \to +\infty}} \sum_{\eta \in E} \sum_{\|d\gamma\| \leqslant CR} d_\eta \lambda^{-\dim V - 1} \\ &\leqslant C_3 \overline{\lim_{\lambda \to +\infty}} (\lambda R)^{\dim V} \lambda^{-\dim V - 1} \\ &= 0.\end{aligned}$$

Conclusion of Proof of Theorem (2.2)

It follows from Lemmas (5.6) and (5.7) that it will be enough to establish the existence of a constant B such that

$$\lim_{n \to +\infty} |I_3^{(\lambda_n)}| \leqslant B\|f\|_p \|g\|_{p'} \tag{23}$$

for all pairs of functions f and g of the form stated in Lemma (5.6). The sequence $\{\lambda_n\}$ is as in the statement of Theorem (2.2). Now

$$\lim_{n \to +\infty} |I_3^{(\lambda_n)}| \leqslant \overline{\lim_{n \to +\infty}} \operatorname{vol}(G/K)^2 \lambda_n^{\dim V} \||\Phi^{(\lambda_n)}\||_p \|S_{\lambda_n R}(f_\lambda)\|_p \|g_{\lambda_n}\|_{p'}, \tag{24}$$

where we have written

$$S_{\lambda_n R}(f_\lambda) = \sum d_\sigma S_{\lambda_n R}(\sigma) \operatorname{Tr}\{\hat{f}_{\lambda_n}(\sigma) \sigma\}.$$

But it follows from the Clerc summability theorem and the assumptions of Theorem (2.2) that the right side of (24) is bounded by a multiple of

$$\overline{\lim_{n \to +\infty}} \lambda_n^{\dim V} M\|f_{\lambda_n}\|_p \|g_{\lambda_n}\|_{p'}. \tag{25}$$

On the other hand, Lemma (3.3)(ii) implies that

$$\|f_{\lambda_n}\|_p \sim \lambda_n^{-(\dim V)/p} \|f\|_p,$$

with a similar statement for $\|g_{\lambda_n}\|_{p'}$. So (23) follows from (24) and (25).

Acknowledgments

We are grateful to Christopher Meaney, John Rice, and Roger Richardson for helpful comments concerning a number of aspects of this paper.

References

1. A. Bonami and J.-L. Clerc, Sommes de Cesaro et multiplicateurs des développements en harmoniques sphériques, *Trans. Amer. Math. Soc.* **183** (1973), 223–263.
2. J.-L. Clerc, Sommes de Riesz et multiplicateurs sur un groupe de Lie compact, *Ann. Inst. Fourier* (*Grenoble*) **24** (1974), 149–172.

3. J. -L. CLERC, Une formule asymptotique du type Mehler–Heine pour les zonales d'un espace riemannien symétrique, *Studia Math.* **57** (1976), 27–32.
4. K. DELEEUW, On L^p multipliers, *Ann. of Math.* **81** (1965), 364–379.
5. A. H. DOOLEY, Contractions of Lie groups and applications to analysis, *in* "Topics in Modern Harmonic Analysis," Ist. Nazionale di Alta Matematica, Rome, 1982.
6. A. H. DOOLEY AND G. I. GAUDRY, An extension of deLeeuw's theorem to the n-dimensional rotation group, *Ann. Inst. Fourier (Grenoble)* **34** (1984), 111–135.
7. A. H. DOOLEY AND J. W. RICE, On contractions of semisimple Lie groups, *Trans. Amer. Math. Soc.* **289** (1985), 185–202.
8. S. HELGASON, "Differential Geometry and Symmetric Spaces," Academic Press, New York, 1962.
9. E. HEWITT AND K. A. ROSS, "Abstract Harmonic Analysis," Vol. II, Springer-Verlag, New York, 1970.
10. S. IGARI, On the multipliers of Hankel transforms, *Tohôku Math. J.* **24** (1972), 201–206.
11. A. KLEPPNER AND R. L. LIPSMAN, The Plancheral formula for group extensions I, *Ann. Sci. Ecole Norm. Sup.* **5** (1972), 459–516.
12. M. LASSALLE, Séries de Laurent des fonctions holomorphes dans la complexification d'un espace symétrique compact, *Ann. Sci. Ecole Norm. Sup.* **11** (1978), 167–210.
13. G. W. MACKEY, "The Theory of Unitary Group Representations," Univ. of Chicago Press, Chicago, 1976.
14. J. F. PRICE, "Lie groups and Compact Groups," London Math. Soc. Lecture Notes, No. 25, Cambridge Univ. Press, Cambridge, 1977.
15. R. L. RUBIN, Harmonic analysis on the group of rigid motions of the Euclidean plane, *Studia Math.* **57** (1978), 125–141.
16. J.-P. SERRE, "Algèbres de Lie semi-simples complexes," Benjamin, New York, 1966.
17. R. J. STANTON, Mean convergence of Fourier series on compact Lie groups, *Trans. Amer. Math. Soc.* **218** (1976), 61–87.
18. R. S. STRICHARTZ, Multiplier transformations on compact Lie groups and algebras, *Trans. Amer. Math. Soc.* **193** (1975), 99–110.
19. N. R. WALLACH, "Harmonic Analysis on Homogeneous Spaces," Dekker, New York, 1973.
20. F. WARNER, "Foundations of Differential Geometry and Lie groups," Scott, Foresman, Glenview, Ill., 1971.